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A ν-fluid model of homogeneous turbulence subjected to uniform mean distortion

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In two previous papers (Proudman 1970; Dowden 1972) it has been shown that some of the phenomena of turbulence at high Reynolds numbers can be modelled by a suitable chosen member of the class of ν-fluids. These are non-Newtonian fluids all of whose properties depend only on a single dimensional constant whose dimensions are those of viscosity. The purpose of this paper is to construct an equation to model homogeneous turbulence in the presence of a spatially constant rate of deformation in the limit of infinite Reynolds number.

The equation employed is that of a doubly degenerate third-order ν-fluid (in Proudman's classification) in the limit $\nu \rightarrow 0$. In such a fluid the stress tensor S is governed by an equation of the form

$$A\dot{S}\ddot{S}+B\dot{S}^2+Cu'S\dot{S}+D\dot{u}'S^2+Eu'^2S^2 = 0,$$

where $A, B, \ldots, E$ are isotropic tensor constants of the fluid, u' is the total rate of deformation tensor and dots denote time derivatives. A list of properties required of the equation and its solution is proposed, and the most general form of $A, B, \ldots, E$ is given consistent with these requirements. Computed solutions of this equation are compared with the results of experiments on homogeneous turbulence, and are found to agree well with them.

1. Introduction

It has been suggested (Proudman 1970) that some properties of turbulence can be described by means of a suitably chosen member of a class of non-Newtonian fluids, ν-fluids, all of whose properties depend on a single dimensional parameter with the dimensions of viscosity. This idea, and Proudman's development of it, has a generality which is not shared by most other models of turbulence. In particular, it concentrates attention on the most fundamental aspects of turbulence such as energy dissipation, dependence on viscosity, and the negative definiteness of the Reynolds stress tensor; it also provides a framework that can include both homogeneous and inhomogeneous turbulence. The theory is not necessarily incompatible with more detailed explanations of various averaged quantities in mechanistic terms but these explanations take a secondary place.

It is the degree of generality which forms the strength of the theory and entitles it to serious consideration. The main disadvantage of it, however, is illustrated by the particular ν-fluid which Proudman put forward as the simplest that could be considered as a model for turbulence. This ν-fluid contains a large number of constants whose values are not known.

A consideration of the relaxation of stress as a model for the decay of homogeneous turbulence (Dowden 1972), however, showed that, if the model were to have a number of very simple properties consistent with the behaviour of homogeneous turbulence in the absence of a mean velocity gradient, the nine coefficients relevant to the problem at infinite Reynolds number must be expressible in terms of only two constants with significance in the solution, and another two with no significance. The substantial simplification of the equation governing the stress that this result implies leads one to suspect that similar simple requirements in the context of more general problems may also lead to a similar reduction in the number of unknown constants. It is the purpose of this paper to find an appropriate set of requirements when the stress is homogeneous and the rate of deformation tensor is a function of time only, and then to state the consequences of these requirements.

An nth-order ν-fluid is one in which the added stress tensor S (taking the density of the fluid to be unity so that kinematic definitions are adopted everywhere), at a point where the velocity is $\mathbf{u}$, is governed by an equation of the form

$$\frac{D^nS}{Dt^n} = f\left(\mathbf{u}, \frac{D\mathbf{u}}{Dt}, \ldots, \frac{D^n\mathbf{u}}{Dt^n}, S, \frac{DS}{Dt}, \ldots, \frac{D^{n-1}S}{Dt^{n-1}}, \nu\right),$$

where f is a function which is regular at the origin of the space of all the arguments shown and all their space derivatives of any order. In the ν-fluid model of turbulence $-S$ is used to describe the Reynolds stress tensor $\{-\overline{v_iv_j}\}$, where $\mathbf{v}$ is the tubulent fluctuation of the velocity field. Proudman concluded that the principle of Galilean invariance must apply to the ν-fluid model of turbulence and consequently that

$$\mathbf{u}, D\mathbf{u}/Dt, \ldots, D^n\mathbf{u}/Dt^n$$

must not occur explicitly in the function, although their space derivatives may. Further, he required that $S = 0$ should be a possible solution.

The equations which govern fluids of the first and second orders possess solutions in the limit as $\nu \to 0$ which are not acceptable as models for the decay of homogeneous turbulence, while the equation of a third-order ν-fluid is of the form

$$\begin{aligned}\dddot{S} = {} & \nu^{-3}p_1^3S^4 + \nu^{-2}(p_2^3S^2\dot{S} + p_3^3\dot{u}'S^3) \\ & + \nu^{-1}(p_4^3S\ddot{S} + p_5^3\dot{S}^2 + p_6^3S^2S'' + p_7^3SS'^2 + p_8^3u'^2S^2 + p_9^3\ddot{u}'S^2 + p_{10}^3u'S\dot{S}) + O(\nu^0),\end{aligned}$$

where dots denote time derivatives, primes denote space derivatives, $p_1^3, \ldots, p_{10}^3$ are isotropic tensor constants and u' is the rate of deformation tensor with $u'_{ij} = \partial u_i/\partial x_j$.

As in the case of first- and second-order ν-fluids a limiting equation as $\nu \to 0$ is obtained which is, in general, much too simple to describe the behaviour of turbulence. A third-order fluid, however, differs from first- and second-order fluids in that if p_1^3, p_2^3 and p_3^3 are all zero a limiting equation results which, as

Proudman has shown, is sufficiently general to describe at least some of the properties of turbulence. If u' and S depend only on time this limiting equation is

$$p_4^3 S\ddot{S} + p_5^3 \dot{S}^2 + p_8^3 u'^2 S^2 + p_9^3 \dot{u}' S^2 + p_{10}^3 u' S\dot{S} = 0. \tag{1.1}$$

Dowden showed that p_4^3 and p_5^3 must have a special form if the solution of (1.1) is to have the following properties when $u' = 0$.

(i) If S is initially real and positive definite and $\dot{\sigma} < 0$, where $\sigma = \mathrm{tr}(S)$, then S continues to have these properties at all subsequent times.

(ii) σ tends to zero as t tends to infinity and the asymptotic stress is isotropic when condition (i) holds.

(iii) The differential structure of the equation is such that σ is always determined by the equation.

The first two terms of (1.1) must then have the form

$$a_2 \Sigma_0 + a_4 \mathrm{tr}(\Sigma_0) I, \tag{1.2}$$

where
$$\Sigma_0 = \ddot{\sigma} S - \frac{1}{3}\frac{2+r}{1+r}\dot{\sigma}^2 I - \frac{2+r}{1+n}\dot{\sigma}(\dot{S} - \tfrac{1}{3}\dot{\sigma} I)$$

and
$$a_2 + 3a_4 \neq 0 \quad (n > r > -1). \tag{1.3}$$

Since the principal purpose of this investigation is to find ν-fluids which are suitable as models of turbulence, this form for the first two terms will be used here. The first part of (1.3) implies that, if (1.2) is to vanish, the equation which results is equivalent to

$$\Sigma_0 = 0.$$

If, therefore, (1.2) is taken as the first two terms of (1.1) it follows that the full equation can be rewritten as

$$a_2 \Sigma + a_4 \mathrm{tr}(\Sigma) I = 0, \tag{1.4}$$

where
$$\Sigma = \Sigma_0 + F(u', \dot{u}', S, \dot{S})$$

and the function F is of the same form as the last three terms on the left-hand side of (1.1).

Write

$$u' = e + \omega,$$

where e is the symmetric and ω the antisymmetric part of u'; the components of ω are then related to the vorticity by

$$\omega_{ij} = -\tfrac{1}{2}\epsilon_{ijk}(\nabla \times \mathbf{u})_k,$$

and e is the rate of strain tensor.

With this notation the general form of Σ is

$$\ddot{\sigma}S-\frac{1}{3}\frac{2+r}{1+r}\dot{\sigma}^2I-\frac{2+r}{1+n}\dot{\sigma}(\dot{S}-\tfrac{1}{3}\dot{\sigma}I)$$
$$+b_1e\sigma\,\mathrm{tr}\,(eS)+b_2e\,\mathrm{tr}\,(eS^2)+b_3e^2\sigma^2+b_4e^2\,\mathrm{tr}\,(S^2)$$
$$+b_5(eS+Se)\,\mathrm{tr}\,(eS)+b_6(e^2S+Se^2)\,\sigma+b_7(e^2S^2+S^2e^2)$$
$$+b_8S^2\,\mathrm{tr}\,(e^2)+b_9S\sigma\,\mathrm{tr}\,(e^2)+b_{10}S\,\mathrm{tr}\,(e^2S)$$
$$+I\{b_{11}\,\mathrm{tr}\,(e^2)\,\mathrm{tr}\,(S^2)+b_{12}\sigma^2\,\mathrm{tr}\,(e^2)+b_{13}\sigma\,\mathrm{tr}\,(e^2S)+b_{14}\,\mathrm{tr}\,(e^2S^2)+b_{15}\,\mathrm{tr}\,(eS)^2\}$$
$$+b_{16}S\sigma\,\mathrm{tr}\,(\omega^2)+b_{17}S\,\mathrm{tr}\,(\omega^2S)+b_{18}S^2\,\mathrm{tr}\,(\omega^2)+b_{19}\omega^2\sigma^2$$
$$+b_{20}\omega^2\,\mathrm{tr}\,(S^2)+b_{21}\sigma(S\omega^2+\omega^2S)+b_{22}(S^2\omega^2+\omega^2S^2)$$
$$+I\{b_{23}\sigma\,\mathrm{tr}\,(S\omega^2)+b_{24}\sigma^2\,\mathrm{tr}\,(\omega^2)+b_{25}\,\mathrm{tr}\,(S^2)\,\mathrm{tr}\,(\omega^2)+b_{26}\,\mathrm{tr}\,(S^2\omega^2)\}$$
$$+I\{b_{27}\,\mathrm{tr}\,(S^2e\omega)+b_{28}\sigma\,\mathrm{tr}\,(Se\omega)\}+b_{29}\,S\,\mathrm{tr}\,(Se\omega)$$
$$+b_{30}\,\mathrm{tr}\,(eS)\,(S\omega-\omega S)+b_{31}\sigma(Se\omega-\omega eS)+b_{32}\sigma(eS\omega-\omega Se)$$
$$+b_{33}\sigma(e\omega S-S\omega e)+b_{34}\,\mathrm{tr}\,(S^2)\,(e\omega-\omega e)+b_{35}\sigma^2(e\omega-\omega e)$$
$$+b_{36}(S^2e\omega-\omega eS^2)+b_{37}(S^2\omega e-e\omega S^2)+b_{38}(eS^2\omega-\omega S^2e)$$
$$+b_{39}\sigma^2\dot{e}+b_{40}\dot{e}\,\mathrm{tr}\,(S^2)+b_{41}\sigma(\dot{e}S+S\dot{e})+b_{42}S\,\mathrm{tr}\,(\dot{e}S)$$
$$+b_{43}(\dot{e}S^2+S^2\dot{e})+I\{b_{44}\sigma\,\mathrm{tr}\,(\dot{e}S)+b_{45}\,\mathrm{tr}\,(\dot{e}S^2)\}$$
$$+b_{46}\sigma(S\dot{\omega}-\dot{\omega}S)+b_{47}(S^2\dot{\omega}-\dot{\omega}S^2)$$
$$+b_{48}\dot{S}\,\mathrm{tr}\,(eS)+b_{49}I\dot{\sigma}\,\mathrm{tr}\,(eS)+b_{50}S\,\mathrm{tr}\,(e\dot{S})+b_{51}I\sigma\,\mathrm{tr}\,(e\dot{S})$$
$$+b_{52}I\,\mathrm{tr}\,(eS\dot{S})+b_{53}e\sigma\dot{\sigma}+b_{54}e\,\mathrm{tr}\,(S\dot{S})+b_{55}\dot{\sigma}(eS+Se)$$
$$+b_{56}\sigma(e\dot{S}+\dot{S}e)+b_{57}(e\dot{S}S+S\dot{S}e)+b_{58}(eS\dot{S}+\dot{S}Se)$$
$$+b_{59}\dot{\sigma}(S\omega-\omega S)+b_{60}\sigma(\dot{S}\omega-\omega\dot{S})+b_{61}(S\dot{S}\omega-\omega S\dot{S})$$
$$+b_{62}(\dot{S}S\omega-\omega S\dot{S})+b_{63}(\dot{S}\omega S-S\omega\dot{S})+b_{64}I\,\mathrm{tr}\,(\dot{S}\omega S). \qquad (1.5)$$

There are a number of additional terms of the correct form (Se^2S for example) but the results obtained by Spencer & Rivlin (1962) show that all such extra terms can be expressed as linear combinations of quantities which already appear in the list, and that this list is minimal can be shown from results obtained by Smith (1965).

As well as (1.4) there are the symmetric and antisymmetric parts of the gradient of the dynamical equation, and the equation of continuity $\mathrm{tr}\,(u')=0$. In the limit as $\nu\to 0$ the dynamical equation becomes

$$\frac{Du_i}{Dt}=-\frac{\partial p}{\partial x_i}+F_i-\frac{\partial S_{ij}}{\partial x_j},$$

where F_i is the body force applied to the system. When S, u' and F' (where F' has (i,j)th component $\partial F_i/\partial x_j$) are homogeneous in space, the symmetric part of the gradient of the dynamical equation is

$$\dot{e}+e^2+\omega^2=-p''+\tfrac{1}{2}(F'+F'^{\mathrm{T}}), \qquad (1.6)$$

where the (i,j)th component of the symmetric tensor p'' is $\partial^2 p/\partial x_i \partial x_j$. The antisymmetric part is

$$\dot{\omega} + e\omega + \omega e = \tfrac{1}{2}(F' - F'^{\mathrm{T}}) \tag{1.7}$$

and the equation of continuity implies that

$$\mathrm{tr}\,(e) = 0.$$

Thus the problem studied is governed by (1.6) and

$$\Sigma = 0. \tag{1.8}$$

The investigation in the absence of a mean velocity field was conducted by imposing the three restrictions quoted earlier on the family of fluids to be investigated; only those which satisfied the restrictions were then considered as possible models for turbulence. The first of these restrictions was derived from the Navier–Stokes equation and the definition of the Reynolds stress tensor, while the second was suggested by experimental evidence. The third was necessary to exclude from consideration members of the family whose differential structure was such that they could not describe the energy of homogeneous turbulence under some circumstances – for example, when the turbulence was isotropic. This method of proposing appropriate restrictions on the family of ν-fluids to be studied and then investigating those members of it which satisfy them is very convenient as it gives rise to definite statements about ν-fluids themselves, as well as directing attention to members of the family most likely to be useful in the study of turbulence. For that reason it will be adopted here.

In the present case little further can be stated in the way of restrictions suggested by theoretical deductions apart from the appropriate generalizations of the first of those used before. There is, however, an additional restriction of the same kind as the third. It will be noticed that the differential structure of (1.8) is the same in general whatever the values of e and ω; that is to say, it takes the form of a system of algebraic equations for $\ddot{\sigma}$ and $\ddot{S} - \frac{1}{3}\ddot{\sigma} I$. There may, however, be particular values of u' for which this algebraic system has more than one solution, or no solution. Any values of e and ω and of S and $\dot{\sigma}$ (provided that S is positive definite and $\dot{\sigma} + 2\,\mathrm{tr}\,(eS) < 0$) must be regarded as acceptable since restrictions on initial conditions deduced from special properties of ν-fluids and not from ideas about turbulence are contrary to the intended generality of the model. Consequently, the values of the constants $b_1, \ldots, b_{64}$ must be such that the equation is always soluble for the highest derivatives. Even a choice of constants for which there is always a unique limit as such special values are approached (by varying e and ω) is not acceptable since the equation as it stands, not a manipulated form, should be adequate for all u'. Further, when ν is small but non-zero the initial behaviour of the solution will depend on ν for these values of e and ω in a way that is qualitatively different from the way in which it will depend for other values.

A preliminary investigation based only on these restrictions shows that although they limit the family of ν-fluids to be studied to some extent (particularly as to acceptable values of $b_{48}, \ldots, b_{64}$) the size of the family is still prohibitively large. While such a large degree of freedom in the choice of models may be desirable or perhaps even essential it would seem more appropriate to impose further restrictions based on experimental evidence. Unfortunately direct evidence of this kind is somewhat scanty and often inconclusive when the results of different sets of experiments on the same phenomenon are compared, so that it is probably more useful to base any new restriction on indirect evidence (either experimental or theoretical) but to choose it so that it has some fairly simple interpretation in terms of the general properties of turbulence. In this way results can be deduced from the study of the appropriate family of ν-fluids which can be compared with experiments and either confirmed or disproved. In either case further understanding of the nature of turbulence will have been gained.

Such a restriction is introduced here, and is the fourth in the list which follows. It is suggested partly by the linearity in u' of the equation for the turbulent velocity perturbations, but a full discussion of it is postponed to the next section. Two substitutions which are convenient in listing these restrictions are the following:

$$\sigma T \equiv S - \tfrac{1}{3}\sigma I, \quad \sigma z \equiv -\dot{\sigma} - 2\,\mathrm{tr}\,(eS). \tag{1.9}$$

When (1.8) is expressed in terms of T and z it becomes a system of first-order equations for these two quantities. If a ν-fluid is used as a model for homogeneous turbulence at infinite Reynolds number z has a particularly simple interpretation since $\tfrac{1}{2}\sigma z$ is the rate of dissipation of energy by the action of viscosity.

The values of $b_1, \ldots, b_{64}$ given in this paper are those for which the equation and its solutions satisfy the following restrictions.

Restriction (i). If S is positive definite and $z > 0$ initially, these conditions remain true subsequently.

Restriction (ii). The equation must be soluble for $\dot{z}$ and $\dot{T}$ whenever S is positive definite and $z > 0$.

Restriction (iii). If u' is identically zero, σ tends to zero as t tends to infinity, and the asymptotic stress is isotropic.

Restriction (iv). $\dot{T}$ and $\dot{z}$ are given by expressions which are linear in the components of u'.

The need for (i) follows at once if $-S$ is to model the Reynolds stress tensor and if the energy is to satisfy the thermodynamic condition. Restriction (iii) ensures that the model has the properties already demanded of it when u' is zero.

2. The fourth restriction

Since the limit $\nu \to 0$ is an unattainable ideal in the flow of a real fluid, it is helpful to have a measure of the circumstances under which this limit in the ν-fluid model might be used as an approximation to a turbulent flow. If

$$R = \sigma/\nu z, \quad \epsilon = z/[\mathrm{tr}\,(u'u'^{\mathrm{T}})]^{\frac{1}{2}}$$

and (as here) we are not primarily concerned with the case $\epsilon \gg 1$, the limit $\nu \to 0$ in the equation of the ν-fluid is a satisfactory approximation if

$$\min\{R, R\epsilon\} \gg 1.$$

Now the length scale of the dissipative turbulent eddies is $(\nu/z)^{\frac{1}{2}}$ while the length scale of convective rotation and distortion due to the mean velocity field is

$$\{\sigma/\mathrm{tr}\,(u'u'^{\mathrm{T}})\}^{\frac{1}{2}}.$$

If ϵ is of order one this is the same as the length scale of the energy-containing eddies, so that the neglect of all but the leading terms in the equation of the ν-fluid is equivalent to assuming that the two different length scales are of different orders of magnitude. This is consistent with Taylor's (1938) observation that a separation occurs between the length scales of the dissipative and energy-containing eddies at high Reynolds numbers. The same interpretation is possible when $\epsilon \gg 1$, but the case $\epsilon \ll 1$ can be obtained in two very different ways. If it is achieved by allowing the magnitude of u' to increase indefinitely while holding everything else fixed, the condition $R\epsilon \gg 1$ is ultimately violated and the turbulence is best described by rapid-distortion theory (Batchelor & Proudman 1954). If on the other hand z is decreased but the magnitude of u' is held fixed, a situation occurs in which the turbulence has the special form demanded by the ν-fluid theory but in which the dissipative eddies are very weak (although their length scales are still small). The mechanics of the development of the turbulence are dominated by the convective terms in the equation of motion, but with the nonlinear terms of greater importance than the viscous terms. No evidence, either experimental or theoretical, appears to be available about the behaviour of turbulence under these conditions, although it is reasonable to assume that the state is unstable.

In turbulence in which the length scales of the dissipative eddies are small compared with those of the energy-containing eddies, the time scales similarly differ by an order of magnitude. It is possible therefore to consider a change in the mean velocity field (described by u') which occurs on a time scale that is short compared with the time scale of the energy-containing eddies but long compared with that of the dissipative eddies. Equations (1.6) and (1.7) show that this is possible in principle by applying an appropriate body force, which will not necessarily be irrotational but whose gradient in space is a function of time only. The energy-containing eddies cannot be affected immediately since their own time scale is too long and there is no time scale associated with the body force other than that of its application, while according to the hypothesis of universal equilibrium (Batchelor 1953, p. 114) the behaviour of the dissipative eddies is uniquely determined statistically by σ, z and ν. Even dependence on T can be included, as may possibly be implied by Townsend's (1954) experiments. The change in u' can directly affect eddies whose time scale lies between the two extremes, but if we assume that the dynamical importance of this is negligible at infinite Reynolds number the effect can be ignored. Now because of the differential structure of the ν-fluid model, σ, z and T are all continuous quantities on the time scale of the energy-containing eddies, so that we would not expect an immediate variation (on the time scale of the variation in u') in any quantity that is the average of products of components of the velocity, their integrals and derivatives. On this assumption the ν-fluid models of such terms will not depend on u, nor by a similar argument, on $\dot{u}'$. Further, they will not depend on $\dot{\sigma}$, $\dot{z}$ or $\dot{T}$, which are given by the constitutive equation of the ν-fluid in terms of u' and hence would, implicitly, give models that could vary on this intermediate time scale. While these arguments are not conclusive, they suggest that it may be of some interest to consider models of such quantities that depend only on z, σ and the components of T; a somewhat similar idea was used by Lumley (1970) to construct a differential equation for the Reynolds stress tensor.

Now the Navier–Stokes equations for a fluid of unit density moving with mean velocity $\mathbf{u}$ in the presence of a body force F show that the turbulent variations $\mathbf{v}$ from the mean velocity satisfy

$$\dot{v}_i + v_j u_{i,j} + u_j v_{i,j} + (v_i v_j - \overline{v_i v_j})_{,j} = -\varpi_{,i} + \nu v_{i,jj}, \quad v_{j,j} = 0, \tag{2.1}$$

where the overbar indicates an average, ϖ is the pressure fluctuation and subscripts preceded by a comma indicate differentiation. If the turbulence is homogeneous, so that the space gradients of all averaged quantities other than $\mathbf{u}$ and the pressure p vanish, (2.1) can be used to show the following results:

$$\mathring{S} + u'S + Su'^{\mathrm{T}} = -\sigma(\Pi + Z), \tag{2.2}$$

$$\Pi_{ij} = R^0_{ijmn} u'_{mn} + R^1_{ij} \tag{2.3}$$

and

$$\tfrac{1}{2} d(\sigma z)/dt + \sigma u'_{ij} R^2_{ij} + \sigma R^3 = 0, \tag{2.4}$$

where

$$S_{ij} = \overline{v_i v_j}, \quad u'_{ij} = u_{i,j}, \quad \sigma = \mathrm{tr}\,(S);$$

$$\sigma \Pi_{ij} = -\overline{\varpi(v_{i,j} + v_{j,i})};$$

$$\sigma Z_{ij} = 2\nu\overline{v_{i,k} v_{j,k}}, \quad z = \mathrm{tr}\,(Z);$$

and

$$\left.\begin{aligned} \sigma R^0_{ijmn} &= 2\overline{(v_{i,j} + v_{j,i})\nabla^{-2} v_{n,m}}, \\ \sigma R^1_{ij} &= \overline{(v_{i,j} + v_{j,i})\nabla^{-2}(v_{m,n} v_{n,m})}, \\ \sigma R^2_{ij} &= 2\nu(\overline{v_{i,k} v_{j,k}} + \overline{v_{k,i} v_{k,j}}), \\ \sigma R^3 &= 2\nu\overline{(v_{i,k} v_k)_{,j} v_{i,j}} + 2\nu^2 \overline{v^2_{i,jk}}. \end{aligned}\right\} \tag{2.5}$$

The equations which govern u' are

$$\dot{u} + u'^2 = -p'' + F', \quad \mathrm{tr}\,(u') = 0, \tag{2.6}$$

where $p'' = \{p_{,ij}\}$ and $F' = \{F_{i,j}\}$. The equations are written on the assumption that F and u depend only on time. Writing as before

$$\sigma T \equiv S - \tfrac{1}{3}\sigma I$$

(2.2) and (2.4) become first-order equations for T and z respectively in terms of Z and $R^0, \ldots, R^3$. The definitions of these latter quantities are given in (2.5) and they are of the type described earlier, so that models for them in terms of quantities defined in the ν-fluid model will, by the argument given above, depend only on σ, z and T. On dimensional grounds σ can be removed from this list, while R^0 is dimensionless, Z, R^1 and R^2 are proportional to z, and R^3 is proportional to z^2. Consequently (2.2) and (2.4) become equations of the form

$$\mathring{T} = zF_1(T) + F_2(u', T)$$

and

$$\dot{z} = z^2 f_1(T) + z f_2(u', T),$$

where F_1 and F_2 are isotropic tensor functions of their arguments and f_1 and f_2 are scalar functions; F_2 and f_2 are linear in the components of u'.

It is this which suggests the form of the fourth restriction given in the previous section.

3. Consequences of the restrictions

If (1.4) and its solutions are to satisfy the restrictions listed at the end of §1, the values of $b_1, \ldots, b_{64}$ must be chosen to be compatible with these constraints. It is possible to investigate these consequences in three stages, corresponding to three distinct situations of physical interest. In the first e is constant with ω zero, in the second ω is constant but e is zero, and in the third both e and ω are functions of time. This investigation is reasonably straightforward but somewhat lengthy, and an outline of the algebra can be found in a report which may be obtained from the author. The conclusion is that Σ must have the form

$$\begin{aligned} &S\{d(\dot{\sigma}+2\,\mathrm{tr}\,(eS))/dt+2[\mathrm{tr}\,(e\dot{S})+2(\mathrm{tr}\,(e^2S)\,\lambda_1-\mathrm{tr}\,(Se\omega)\,\lambda_2)\,(n-r)/(r+1)] \\ &\quad\times(k-r-1)\,/(n+1)\}-\{\dot{S}-\tfrac{1}{3}(\dot{\sigma}+2\,\mathrm{tr}\,(eS))\,I+[\lambda_1(eS+Se)+\lambda_2(S\omega-\omega S)] \\ &\quad\times(n-r)/(r+1)\}\,(\dot{\sigma}+2\,\mathrm{tr}\,(eS))\,(r+2)/(n+1)-\tfrac{1}{3}I(\dot{\sigma}+2\,\mathrm{tr}\,(eS))^2\,(r+2)/(r+1) \\ &\quad+b_{60}\{\sigma(\dot{S}\omega-\omega\dot{S})-(S\omega-\omega S)\,(\dot{\sigma}+2\,\mathrm{tr}\,(eS))\,(n+1)/(r+1)+[\lambda_1(\sigma(Se\omega-\omega eS \\ &\quad+eS\omega-\omega Se)-2(S\omega-\omega S)\,\mathrm{tr}\,(eS))+\lambda_2\sigma(S\omega^2+\omega^2S-2\omega S\omega)]\,(n-r)/(r+1)\} \\ &\quad+b_{64}I\{\mathrm{tr}\,(\dot{S}\omega S)+[\mathrm{tr}\,(S^2e\omega)\,\lambda_1+(\mathrm{tr}\,(S^2\omega^2)-\mathrm{tr}\,(S\omega S\omega))\,\lambda_2]\,(n-r)/(r+1)\}, \end{aligned} \quad (3.1)$$

where b_{60}, b_{64}, k, λ_1, λ_2, n and r are constants such that

$$\left.\begin{aligned} b_{60}b_{64} = (k-r-1)\,\{\lambda_1-(r+1)/(n-r)\} &= 0 \\ \text{and}\qquad n > r > -1. & \end{aligned}\right\} \quad (3.2)$$

In fact b_{60} and b_{64} have no significance in the solutions of (1.4), so that these solutions depend only on four parameters, n, r, λ_2 and either λ_1 or k.

A further condition that must be satisfied if the ν-fluid is to model turbulence is

$$\lambda_1 \leqslant (r+1)/(n-r); \quad (3.3)$$

this is not a consequence of restrictions (i)–(iv) but follows from the form of R^0, defined in (2.5).

The relative complexity of (3.1) obscures the remarkable simplicity of the result which becomes apparent on using the substitutions (1.9) for S and $\dot{\sigma}$. Equation (1.4) then reduces to the following pair of equations for z and T:

$$\begin{aligned} &\dot{T}+\{\lambda_1(eT+Te-\tfrac{2}{3}\,\mathrm{tr}\,(eT)\,I+\tfrac{2}{3}e-2\,\mathrm{tr}\,(eT)\,T)+\lambda_2(T\omega-\omega T)+zT\} \\ &\qquad\qquad\times(n-r)/(r+1) = 0 \end{aligned} \quad (3.4)$$

$$\text{and}\quad \dot{z}+\frac{z^2}{r+1}+z\,\mathrm{tr}\,(eT)\left\{\frac{2(k-r-1)}{r+1}-2\left(\lambda_1-\frac{r+1}{n-r}\right)\frac{(n-r)\,(r+2)}{(r+1)\,(n+1)}-2\right\} = 0. \quad (3.5)$$

In general, it is necessary to solve these equations numerically; however, when ω is constant and e is zero, an explicit solution exists for an initial-value problem posed at $t = 0$. Distinguishing initial values by the subscript zero, the solution is

$$\left.\begin{aligned} z &= z_0/\{1+z_0t/(r+1)\}, \quad \sigma = \sigma_0/\{1+z_0t/(r+1)\}^{r+1} \\ \text{and} \qquad T &= \frac{\exp\{\lambda_2\omega t(n-r)/(r+1)\}T_0\exp\{-\lambda_2\omega t(n-r)/(r+1)\}}{\{1+z_0t/(r+1)\}^{n-r}}. \end{aligned}\right\} \tag{3.6}$$

It appears from this that σ is unaffected by the superposition of a rigid-body rotation and that if λ_2 is non-zero the amplitude of T decreases slowly according to a power law, but superimposed on this is an oscillatory pattern of behaviour in some of its components. This solution, in fact, implies that there is a co-ordinate system which rotates about the same axis as the fluid but at $\lambda_2(n-r)/(r+1)$ times the rate in which the stress appears to relax exactly as if the fluid and the co-ordinate system were both at rest.

4. Comparison with experimental results

Solutions of (1.9), (3.4) and (3.5) have been calculated numerically for direct comparison with the results obtained in a number of experiments on the behaviour of homogeneous turbulence in the presence of a mean velocity gradient. The experiments are of two types. First are the experiments on turbulence in the presence of a plane strain carried out by Townsend (1954), Maréchal (1967) and Tucker & Reynolds (1968). The second set of experiments consists of those of Rose (1966) and Champagne, Harris & Corrsin (1970) on the effect of a uniform shear. Some of the parameters of these experiments are given in table 1.

		Type	u' (s^{-1})†	ϵ	R	$R\epsilon$	Grid Rey-nold's no.
Townsend	1954	Plane strain	$-u'_{11} = u'_{22} = 9{\cdot}4$	0·9	230	210	12000
Maréchal	1967	Plane strain	$-u'_{11} = u'_{22} = 18{\cdot}9$	0·5	640	320	20000
Tucker & Reynolds	1968	Plane strain	$-u'_{11} = u'_{22} = 4.45$	2·1	400	840	12000
Rose	1966	Uniform shear	$u'_{12} = 13{\cdot}6$	3·1	400	1240	9000
Champagne *et al.*	1970	Uniform shear	$u'_{12} = 12{\cdot}9$	0·8	950	760	17000
Traugott	1958	Rigid rotation	$u'_{21} = -u'_{12} = 209$	0·4	280	110	—

† The remaining components of u' are zero.

TABLE 1

A preliminary set of calculations showed that it is not possible to obtain anything approaching numerical agreement with the plane strain experiments for any values of n, r and k if

$$\lambda_1 = (r+1)/(n-r),$$

and so all further calculations were carried out with

$$k = r+1,$$

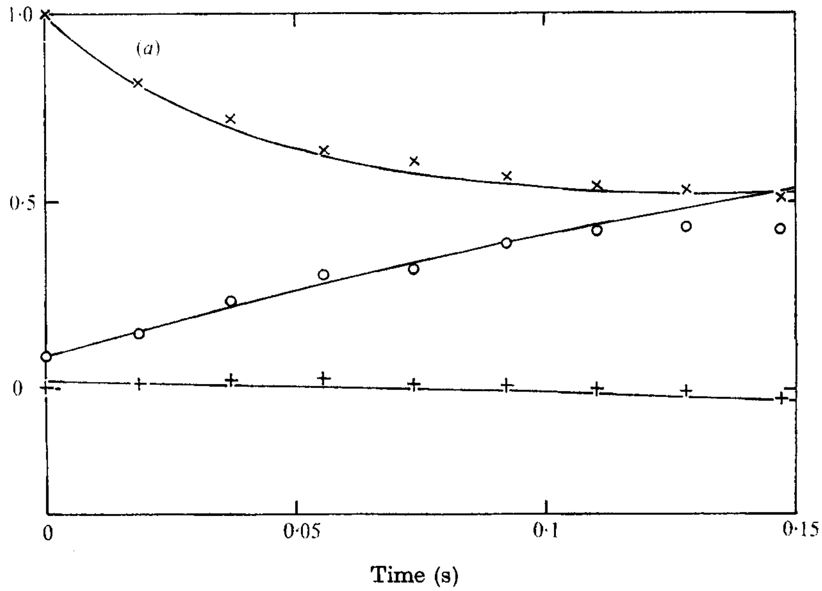
leaving λ_1 as a parameter to be chosen arbitrarily subject to (3.3). The value $r = 0$ was always found to give rather better agreement of the curve for the turbulent energy with the experimental points, a result which can be seen in figure 6 of Tucker & Reynolds' paper. The curves shown here have all been computed with this value, and with $n = 0{\cdot}33$, which follows from the figure for $n-r$ suggested by the experimental evidence on the decay of homogeneous turbulence in the absence of a mean velocity gradient (Dowden 1972). Attempts at varying this value for n showed that, while agreement with some experiments might be somewhat improved, that for others was usually made worse. It is possible that a detailed study may reveal a better value but this one is quite adequate for our present purpose. The value $\lambda_1 = 0{\cdot}7$ was also obtained by trial and error; again, it may be possible to improve on it somewhat, but the agreement of the solutions calculated from these values with the experimental observations is very encouraging as can be seen from figure 1. The initial values used in the calculation were chosen to give the best possible agreement in the middle section of the wind tunnel but it will be noticed that towards the end in all cases there are discrepancies. Lumley has pointed out that the results of the experiments differ substantially from each other and has suggested that these differences may be due to end effects. It can be seen from the curves given here that some such explanation is again necessary but that the agreement is substantially better than that produced by Lumley's model, which (though similar) was somewhat simpler than the one presented here.

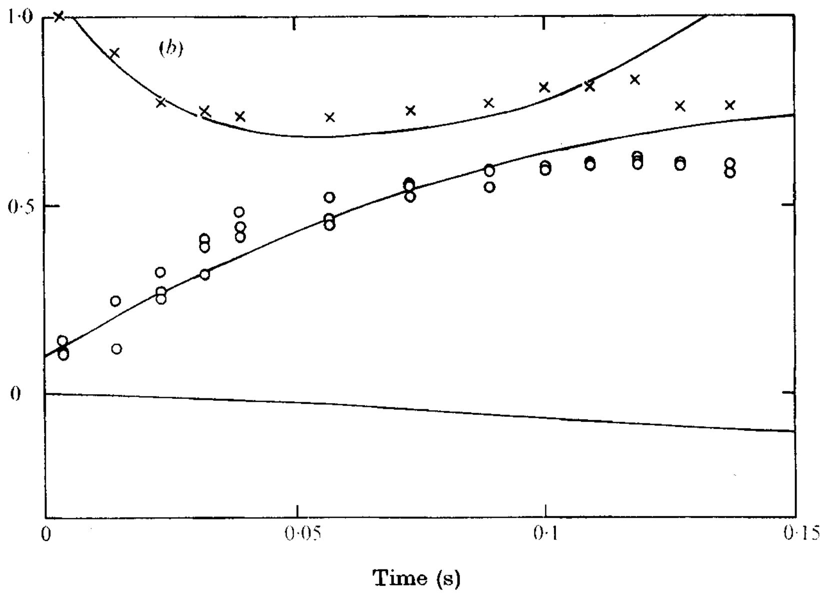
With values for n, r and λ_1 known it is possible to compute solutions for comparison with the uniform shear experiments. It was found that they were relatively insensitive on this time scale to the value chosen for λ_2, but the value finally chosen, for reasons to be discussed later, was $\lambda_2 = 0{\cdot}978$, and the results of the calculations are shown in figure 2. Once again the agreement with the experimental results is good, especially with the experiments of Champagne *et al.* The agreement with Rose's observations is less satisfactory, particularly for the energy and cross-correlation terms. This may be due to the fact that, although the observations shown were taken on the centre-line of the tunnel, there was substantial variation in the cross-correlation term across the tunnel. The range of variation in the central region of the tunnel is also shown, and it can be seen that the computed solution is compatible with it.

One solution of each type was computed over a much greater period of time than that covered by the experiments to illustrate the asymptotic behaviour of the solutions, and the results are shown in figures 3 (*a*) and (*b*). It is quite clear from this that the time scale of the experiments in all cases was far too short to support or contradict these predictions. What effect a finite Reynolds number may have over such long periods of time is also an open question. However the predictions are as follows.

In plane strain an asymptotic structure is attained but the energy of the turbulence increases without limit.

In uniform shear the asymptotic behaviour depends critically on the value of λ_2. For λ_2 less than $0{\cdot}978$ the behaviour is qualitatively similar to that for plane strain; for λ_2 greater than this value the energy of the turbulence has an overall





FIGURES (1 *a*, *b*). For legend see facing page.

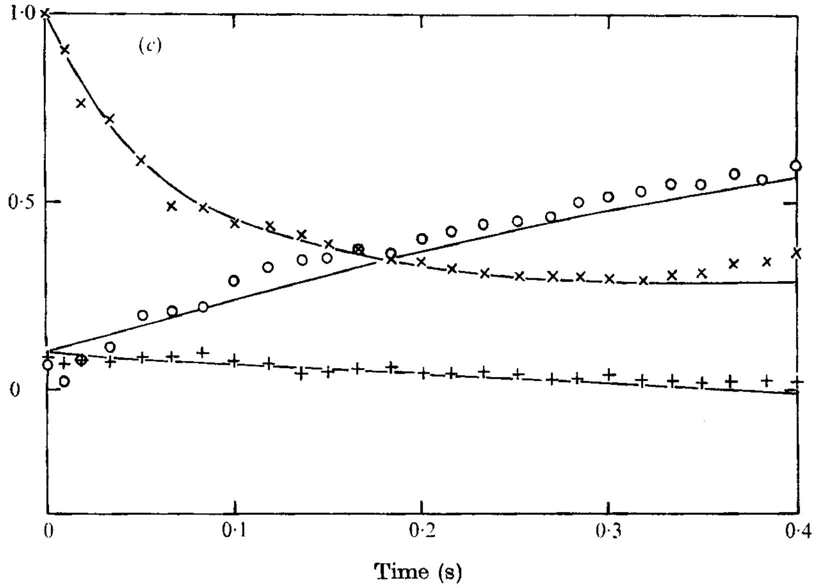


FIGURE 1. Computed solutions of (5.3) compared with observations of the effect of a plane strain on homogeneous turbulence. Values of constants in the equation used in the calculation were $r = 0$, $n = 0{\cdot}33$, $\lambda_1 = 0{\cdot}7$ and $\lambda_2 = 0{\cdot}978$. Observations: $\times$, σ/σ_0; $+$, T_{33}; $\bigcirc$, $K_1 \equiv (T_{11}-T_{22})/(T_{11}+T_{22}+\frac{1}{3})$. (*a*) Townsend (1954, 1 in. grid). (*b*) Maréchal (1967; the values for σ/σ_0 have been reconstructed on the assumption that $T_{33} \equiv 0$). (*c*) Tucker & Reynolds (1968).

tendency to decrease while oscillations (with a period of the order of 1 s in the case studied) are superimposed on it and on all the components of the stress tensor, and the cross-correlation tends to zero; for λ_2 with the value given here, both an equilibrium structure and a non-zero equilibrium value for the turbulent energy are attained but with a zero value which is reached very slowly for the cross-correlation. It is remarkable that the equilibrium value is not that suggested by the experiment, and indeed this latter value is seen to be only a local minimum with a time scale which happens to be of the same order of magnitude as the time scale of the experiment itself. From (3.4) and (3.5) it can be seen that solutions for the family of mean flows in which

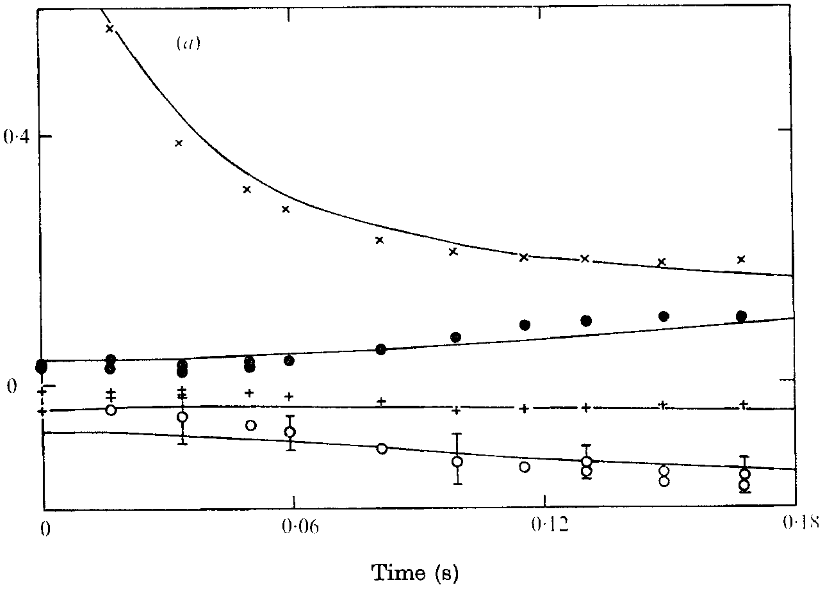
$$e\omega + \omega e = 0$$

can be expressed in terms of the parameter

$$\lambda_2 |\mathrm{tr}\,(\omega^2)/\mathrm{tr}\,(e^2)|^{\frac{1}{2}}$$

for given values of n, r, λ_1 and e. Uniform shear occurs when this parameter is equal to λ_2 and is the special case which separates flows with closed streamlines from flows with open streamlines. From (1.9) equilibrium of the energy with a non-zero value can only occur if a steady state is possible in which

$$z + 2\,\mathrm{tr}\,(eT) = 0.$$



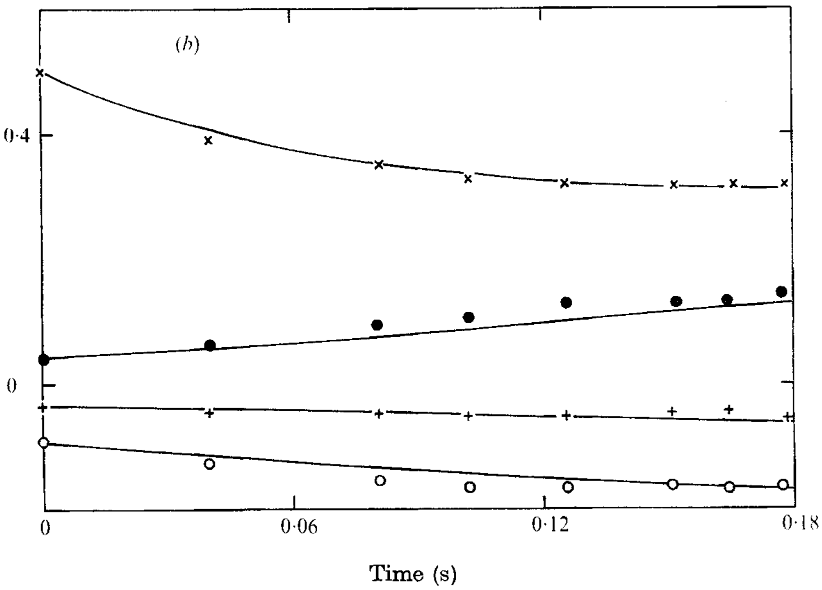
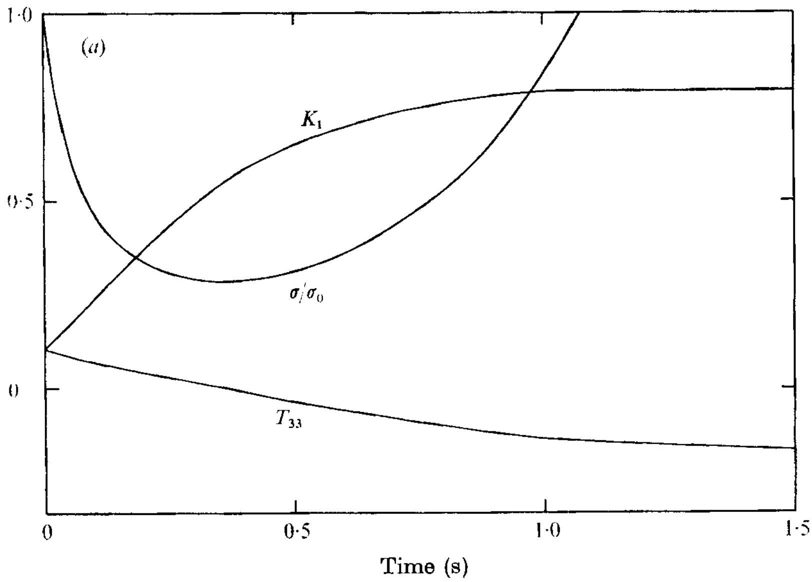
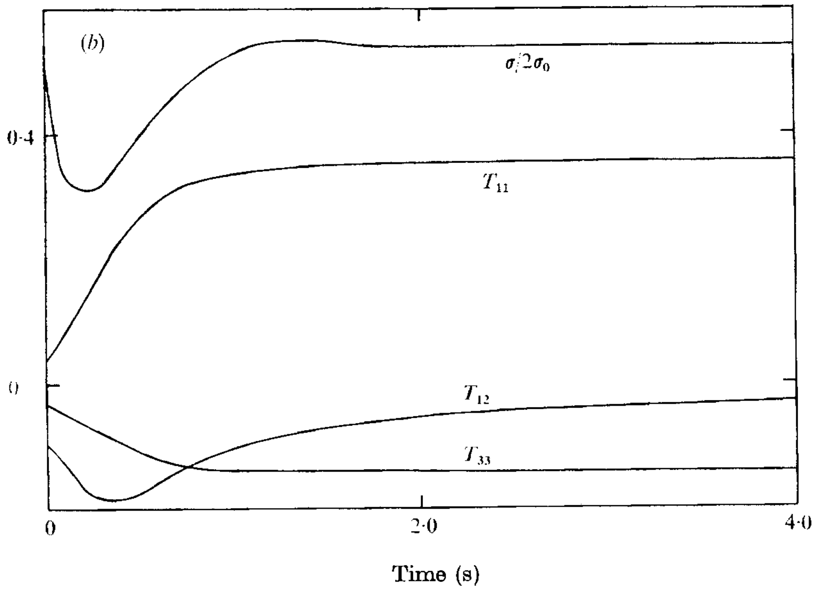


FIGURE 2. Computed solutions of (5.3) compared with observations of the effect of a uniform shear on homogeneous turbulence. The constants have the same values as for figure 1. Observations: ●, T_{11}; +, T_{33}; ○, T_{12}. (a) Rose (1966). The range of variation in T_{12} across the central portion of the tunnel is shown by error bars. Observations: ×, σ/σ_0. (b) Champagne *et al.* (1970). Observations: ×, $\sigma/2\sigma_0$.





FIGURES 3 (*a*, *b*). For legend see next page.

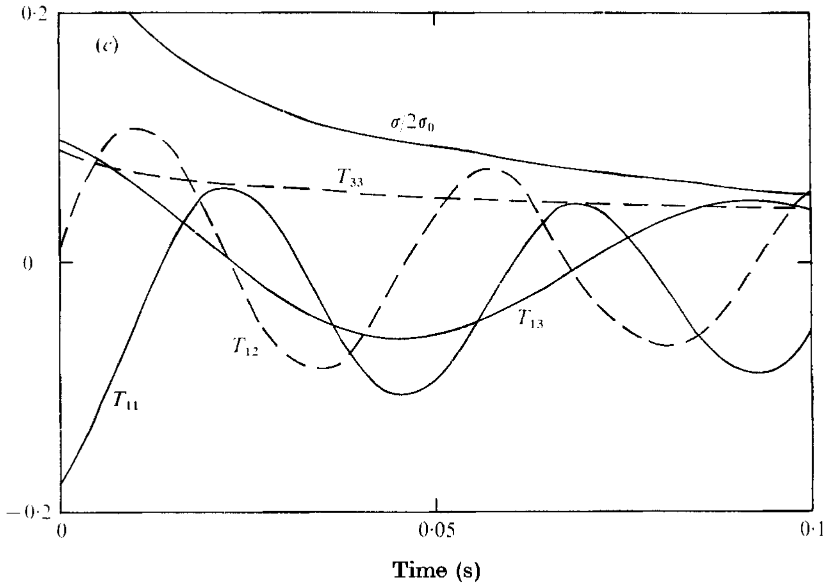


FIGURE 3. Computed solutions of (5.3) calculated over longer periods of time to show asymptotic behaviour. (a) Data as for figure 1(c) (plane strain). (b) Data as for figure 2(b) (uniform shear). (c) Data as for Traugott (1958; rigid rotation).

The calculated solutions suggest that T_{12} (in the co-ordinate system indicated in table 1) tends to its asymptotic value more slowly than σ, T_{11} or T_{22}. If this is so then (3.4) can be used to find T on the assumption that T_{12} is non-zero. For consistency with (3.5), however, the asymptotic value of T_{12} calculated in this way must itself be zero and this gives rise to the condition

$$\lambda_2^2 \operatorname{tr}(\omega^2) + (\lambda_1^2 + \tfrac{2}{3}\lambda_1) \operatorname{tr}(e^2) = 0.$$

This relation therefore gives the ratio of the magnitudes of ω and e at which a change in the character of the solutions occurs. There is no clearcut reason for preferring one value of this ratio to any other, but the only value which has special physical significance occurs when

$$\operatorname{tr}(\omega^2) + \operatorname{tr}(e^2) = 0,$$

as pointed out earlier. This makes it plausible to choose λ_1 and λ_2 so that

$$\lambda_2^2 = \lambda_1^2 + \tfrac{2}{3}\lambda_1,$$

and it is for this reason that a value of 0·978 was chosen for λ_2 in the calculated solutions.

Experimental evidence on the behaviour of homogeneous turbulence in the presence of (for example) a rigid-body rotation would be of very great interest. This model predicts that in a stationary frame of reference the turbulent energy should decay as if no rotation were present, and that the asymptotic stress is isotropic, but that T has periodic fluctuations superimposed on this behaviour.

An experiment on the effect of a rigid-body rotation on turbulence was carried out by Traugott (1958). The turbulence he observed was almost isotropic and appeared to decay in much the same way as when no rotation was present, so that his observations support the predictions of (3.6). His apparatus however was axisymmetric (as well as rather short) and so his observations cannot be used to test predictions about the structure of S. The data for his experiments are included in table 1 and the solution for an anisotropic initial-value problem with these data is shown in figure 3(*c*).

5. Discussion

The restrictions set out in §1 can be used to find the most general doubly degenerate ν-fluid which satisfies them in the limit as $\nu \to 0$. Although this member of the family depends on remarkably few constants it is still possible to choose them so that all the direct experimental evidence is surprisingly well described, even though the asymptotic behaviour of the solutions is not always that which the experiments appear to imply. Although tests of the arguments employed in §2 to suggest our fourth restriction would be of interest in their own right this model depends only on the restriction itself and not directly on the arguments used to support it. Consequently, the main tests of the model must be the comparison of its predictions in as wide a range of circumstances as possible with experimental results. The available evidence to date cannot be regarded as sufficient to provide a rigorous test of the model, however, and further experiments on, for example, the behaviour of homogeneous turbulence in the presence of a rigid-body rotation would be of great interest.

Finally, it is perhaps advisable to point out that if (3.1) should prove to be inadequate as a model of homogeneous turbulence in the light of further evidence, this would not necessarily invalidate the concept of the ν-fluid model nor would it necessarily lead to the rejection of all doubly degenerate third-order models.

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